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On A.E. Convergence of Durrmeyer–Stieltjes Polynomials*

L. SZILI

Department of Numerical Analysis, Eötvös L. University, H-1088 Budapest, Múzeum krt. 6–8., Hungary

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Let ν be a finite Borel measure on $[0,1]$. We introduce the notation of the Durrmeyer–Stieltjes polynomials

$$D_n\nu = (n+1)\sum_{k=0}^{n}\left(\int_0^1 N_{k,n}\,d\nu\right)N_{k,n},$$

where $N_{k,n}(x) = \binom{n}{k}x^k(1-x)^{n-k}$ $(x \in [0,1],\ k = 1,2,\dots,n)$ are the basic Bernstein polynomials. We prove that the maximal operator of the sequence (D_n) is of weak type and the sequence of polynomials $(D_n\nu)$ converges a.e. on $[0,1]$ to the absolutely continuous part of ν. © 1994 Academic Press, Inc.

1. INTRODUCTION

Let $n \in \mathbb{N}$ be a natural number and denote $\mathscr{P}_n$ the $(n+1)$-dimensional space of algebraic polynomials of degree at most n. Let $L^0 = L^0[0,1]$ represent the collection of a.e. finite, Lebesgue measurable functions and denote by $|A|$ the Lebesgue measure of a set $A \subseteq [0,1]$. The space $L^1 = L^1[0,1]$ is considered as a real Banach space of real-valued functions with the usual norm

$$\|f\|_1 := \int_0^1 |f(t)|\,dt, \qquad f \in L^1.$$

J. L. Durrmeyer [7] introduced the following modification of the classical Bernstein polynomial operators,

$$D_n\colon L^1 \to \mathscr{P}_n, \qquad D_nf := (n+1)\sum_{k=0}^{n}\left(\int_0^1 N_{k,n}(t)f(t)\,dt\right)N_{k,n}\ (n \in \mathbb{N}), \tag{1}$$

*Research supported by the National Scientific Research Foundation (Grant 384/324/0413). E-mail address: SZILI@LUDENS.ELTE.HU.

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where

$$N_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k} \qquad (x \in [0,1],\ k = 0,1,\dots,n)$$

denotes the basic Bernstein polynomials of degree n. In [7] Durrmeyer proved that for every continuous function $f \in C[0,1]$ the sequence of polynomials $D_n f$ ($n \in \mathbb{N}$) uniformly tends to f on the interval $[0,1]$.

Further interesting properties of the sequence of these operators were studied by M. M. Derrienic [5], Z. Ciesielski [3], Z. Ditzian and K. Ivanov [6], and other authors (see [8–10, 12, 13, 17]). For the case of a.e. convergence M. M. Derrienic [5] proved the following result.

THEOREM A. *For every function $f \in L^1$ the sequence of polynomials $(D_n f)_{n \in \mathbb{N}}$ converges a.e. to f on* $[0,1]$.

In this paper we shall prove a generalization of this result to finite Borel measures.

Let $\mathbb{M}$ denote the collection of finite Borel measures on $[0,1]$ and $\|\nu\|$ the total variation of the measure $\nu \in \mathbb{M}$. The *maximal function* of a measure $\nu \in \mathbb{M}$ at a point $x \in [0,1]$ is defined by

$$\nu^*(x) := \sup \frac{|\nu(I)|}{|I|},$$

where the supremum is taken over all intervals I contained in $[0,1]$ and containing x.

It is known (see [14]) that for every measure $\nu \in \mathbb{M}$ the following inequality holds

$$|\{x \in [0,1]\colon \nu^*(x) > y\}| \leqq \frac{5}{y}\|\nu\| \tag{2}$$

for all $y > 0$, i.e., the maximal operator

$$M\colon \mathbb{M} \to L^0, \qquad M\nu := \nu^*$$

is of weak type.

Recall that if $\nu \in \mathbb{M}$ is an absolutely continuous measure, then its Radon–Nikodym derivative (which we shall denote by $d\nu/dm$) with respect to the Lebesgue measure m exists and

$$\nu(A) = \int_A \frac{d\nu}{dm} \qquad (A \subset [0,1]).$$

It is also known that for every finite Borel measure ν there exists a uniquely determined absolutely continuous measure ν_f and a singular measure λ such that

$$\nu = \nu_f + \lambda .$$

Such a measure ν_f is called *the absolutely continuous part of* $\nu \in \mathbb{M}$.

2. Main Results

We introduce the notation of the so-called *Durrmeyer–Stieltjes operators*, as

$$D_n : \mathbb{M} \to \mathscr{P}_n , \qquad D_n \nu := (n+1) \sum_{k=0}^{n} \left(\int_0^1 N_{k,n} \, d\nu \right) N_{k,n} \; (n \in \mathbb{N}) . \quad (3)$$

Another generalization of the polynomials (1) have been introduced and investigated by Z. Ciesielski [4] and H. Berens and Y. Xu [1, 2].

The *maximal operator* of the sequence of the Durrmeyer–Stieltjes operators (3) will be defined by

$$(D^* \nu)(x) := \sup_{n \in \mathbb{N}} |D_n \nu(x)| \qquad (x \in [0,1];\ \nu \in \mathbb{M}) .$$

The aim of this note is to prove the following statements.

THEOREM 1. *For every measure* $\nu \in \mathbb{M}$ *the following inequality is satisfied*

$$(D^* \nu)(x) \leqq (\sqrt{2} + 1) \nu^*(x) \qquad (x \in (0,1)) .$$

THEOREM 2. *Let* $\nu \in \mathbb{M}$ *be a finite Borel measure on the interval* $[0,1]$. *Denote* f *as the Radon–Nikodym derivative of the absolutely continuous part of* ν. *Then the sequence of the Durrmeyer–Stieltjes polynomials* (3) *satisfies the limit relation*

$$\lim_{n \to \infty} D_n(\nu) = f \qquad a.e. \text{ on } [0,1] .$$

Remark. If the measure $\nu \in \mathbb{M}$ is absolutely continuous and its Radon–Nikodym derivative is f then $D_n \nu = D_n f$ $(n \in \mathbb{N})$, so from Theorem 2 we have Theorem A.

3. Auxiliaries

In order to prove the theorems we need some preliminary results and lemmas. We will suppose a function of bounded variation on $[0,1]$ is continuous from the left at all points of $(0, 1]$ and continuous from the right at the point 1 in the sequel.

It is known that for every measure $\nu \in \mathbb{M}$ there exists a function F_ν: $[0, 1] \to \mathbb{R}$ of bounded variation on $[0, 1]$ such that

$$\int_0^1 g \, dF_\nu = \int_0^1 g \, d\nu \tag{4}$$

for all functions g integrable with respect to the measure ν (the space of all these functions is denoted by L_ν^1). The function F_ν with the above property is not uniquely determined. Indeed for every number $c \in \mathbb{R}$ the function $F = F_\nu + c$ satisfies the equality

$$\int_0^1 g \, dF = \int_0^1 g \, d\nu \tag{5}$$

for all $g \in L_\nu^1$.

It is also true that if the functions F_ν, F satisfy (4) and (5) then there exists a real number $c \in \mathbb{R}$ such that $F = F_\nu + c$.

For the proof of the theorems we need some other representation of the Durrmeyer–Stieltjes polynomials.

LEMMA 1. *For every measure* $\nu \in \mathbb{M}$ *the Durrmeyer–Stieltjes polynomials* (3) *can be written in the form*

$$\begin{aligned} D_n\nu(x) = D_n(dF)(x) &= (n+1) \sum_{k=0}^{n} \left(\int_0^1 N_{k,n} \, dF \right) N_{k,n}(x) \\ &= (n+1)\left[F(1)x^n - F(0)(1-x)^n \right] \\ &\quad - n(n+1) \sum_{k=0}^{n-1} \frac{N_{k+1,n+1}(x)}{x(1-x)} \left(x - \frac{k+1}{n+1} \right) \\ &\quad \times \left(\int_0^1 N_{k,n-1}(t) F(t) \, dt \right) \end{aligned} \tag{6}$$

for all $x \in (0, 1)$ *and* $n \in \mathbb{N}$, *where* F: $[0, 1] \to \mathbb{R}$ *is an arbitrary function of bounded variation with property* (5).

Proof. Let $\nu \in \mathbb{M}$ be a fixed measure and denote F as the function of bounded variation with the property (5).

Using integration by parts with respect to the Lebesgue–Stieltjes integral we have for every function F: $[0, 1] \to \mathbb{R}$ of bounded variation

$$\int_0^1 N_{k,n} \, dF + \int_0^1 F \, dN_{k,n} = [N_{k,n} F]_0^1 \qquad (k = 0, 1, \ldots, n;\ n \in \mathbb{N}).$$

Since the basic polynomials $N_{k,n}$ $(k = 0, 1, \ldots, n;\ n \in \mathbb{N})$ are absolutely continuous functions thus

$$\int_0^1 F\, dN_{k,n} = \int_0^1 F(t) N'_{k,n}(t)\, dt \qquad (k = 0, 1, \ldots, n;\ n \in \mathbb{N}).$$

Using the above identities and the relations $N_{k,n}(0) = \delta_{0,k}$ and $N_{k,n}(1) = \delta_{k,n}$ we conclude that

$$\begin{aligned}
D_n(dF)(x) &= (n+1) \sum_{k=0}^{n} \left(\int_0^1 N_{k,n}\, dF \right) N_{k,n}(x) \\
&= (n+1) \sum_{k=0}^{n} [N_{k,n} F]_0^1 N_{k,n}(x) \\
&\qquad - (n+1) \sum_{k=0}^{n} \left(\int_0^1 F(t) N'_{k,n}(t)\, dt \right) N_{k,n}(x) \\
&= (n+1)\left[F(1) x^n - F(0)(1-x)^n \right] \\
&\qquad - (n+1) \sum_{k=0}^{n} \left(\int_0^1 F(t) N'_{k,n}(t)\, dt \right) N_{k,n}(x). \qquad (7)
\end{aligned}$$

From the definition of the basic Bernstein polynomials it follows that

$$N'_{0,n}(t) = -n(1-t)^{n-1}, \qquad N'_{n,n}(t) = nt^{n-1},$$
$$N'_{k,n}(t) = n\left[N_{k-1,n-1}(t) - N_{k,n-1}(t) \right], \qquad \text{if } 1 \leqq k \leqq n-1,$$

thus

$$\begin{aligned}
&(n+1) \sum_{k=0}^{n} \left(\int_0^1 N'_{k,n}(t) F(t)\, dt \right) N_{k,n}(x) \\
&\quad = (n+1) \left[\left(\int_0^1 N'_{0,n}(t) F(t)\, dt \right) N_{0,n}(x) \right. \\
&\qquad\qquad \left. + \left(\int_0^1 N'_{n,n}(t) F(t)\, dt \right) N_{n,n}(x) \right] \\
&\qquad + n(n+1) \sum_{k=1}^{n-1} \left(\int_0^1 \left[N_{k-1,n-1}(t) - N_{k,n-1}(t) \right] F(t)\, dt \right) N_{k,n}(x) \\
&\quad = n(n+1) \sum_{k=0}^{n-1} \left(\int_0^1 N_{k,n-1}(t) F(t)\, dt \right) \left[N_{k+1,n}(x) - N_{k,n}(x) \right].
\end{aligned}$$

An easy calculation shows that for every $x \in (0,1)$

$$N_{k+1,n}(x) - N_{k,n}(x) = \frac{N_{k+1,n+1}(x)}{x(1-x)}\left(x - \frac{k+1}{n+1}\right),$$

from which we obtain that

$$\begin{aligned}(n+1)\sum_{k=0}^{n}\left(\int_0^1 N'_{k,n}(t)F(t)\,dt\right)N_{k,n}(x)\\ = n(n+1)\sum_{k=0}^{n-1}\frac{N_{k+1,n+1}(x)}{x(1-x)}\left(x - \frac{k+1}{n+1}\right)\left(\int_0^1 N_{k,n-1}(t)F(t)\,dt\right).\end{aligned}$$

Combining this with (7) we get the representation (6). ∎

Let us consider the polynomials

$$A_{m,n}(x) := \sum_{k=0}^{n-1} N_{k+1,n+1}(x)\int_0^1 N_{k,n-1}(t)(t-x)^m\,dt \qquad (x \in \mathbb{R};\ m,n \in \mathbb{N}). \quad (8)$$

LEMMA 2. *Let $n \geqq 2$ be an arbitrary integer. Then the following estimates hold*:

$$A_{2,n}(x) \leqq 2\frac{x(1-x)}{n(n+2)} \qquad (x \in [0,1]), \qquad (9)$$

$$A_{4,n}(x) \leqq 9\frac{x(1-x)}{n(n+2)(n+3)} \qquad (x \in [0,1]). \qquad (10)$$

Proof. The polynomials defined by (8) are the same as those introduced by Z. Ditzian and K. Ivanov [6, p. 86] disregarding a factor n. As their polynomials obey a recursion formula [6, p. 87], the same holds for our polynomials:

$$\begin{aligned}x(1-x)\left[A'_{m,n}(x) - mA_{m-1,n}(x)\right]\\ = -(n+m+1)A_{m+1,n}(x) - m(1-2x)A_{m,n}(x)\\ + mx(1-x)A_{m-1,n}(x) \qquad (x \in \mathbb{R};\ m = 1,\dots,n). \quad (11)\end{aligned}$$

Using the well-known relations (see [15])

$$\sum_{k=0}^{n+1} N_{k,n+1}(x) = 1,$$

$$\sum_{k=0}^{n+1} kN_{k,n+1}(x) = (n+1)x \ (x \in [0,1], n \in \mathbb{N}),$$

$$\sum_{k=0}^{n+1}\left(x - \frac{k}{n+1}\right)N_{k,n+1}(x) = 0 \qquad (x \in [0,1], n \in \mathbb{N}),$$

$$\int_0^1 N_{k,n-1}(t)\,dt = \frac{1}{n} \qquad (k = 0,1,\dots,n-1;\, n \in \mathbb{N}\setminus\{0,1\}),$$

$$\int_0^1 tN_{k,n-1}(t)\,dt = \frac{k+1}{n(n+1)} \tag{12}$$

$$(k = 0,1,\dots,n-1;\, n \in \mathbb{N}\setminus\{0,1\})$$

we have

$$A_{0,n}(x) = \frac{1-(1-x)^{n+1}-x^{n+1}}{n} \qquad (x \in [0,1];\, n \in \mathbb{N})$$

and

$$A_{1,n}(x) = \frac{(1-x)x^{n+1}-x(1-x)^{n+1}}{n} \qquad (x \in [0,1];\, n \in \mathbb{N}). \tag{13}$$

Specializing (11) for the case $m = 1$ simple calculations show

$$A_{2,n}(x) = \frac{x(1-x)}{n(n+2)}\left\{2-(n+2)\left[x(1-x)^n+(1-x)x^n\right]\right\}$$

$$(x \in [0,1];\, n \in \mathbb{N}), \tag{14}$$

from which we get the inequality (9).

In order to prove (10), first we calculate $A_{3,n}(x)$ from (11), (12), and (13):

$$A_{3,n}(x) = \frac{x(1-x)}{n}\left\{(1-x)^2x^n - x^2(1-x)^n - 6\frac{1-2x}{(n+2)(n+3)}\right\}. \tag{15}$$

Finally putting $m = 3$ into (11) and using (13), (14) we get

$$A_{4,n}(x) = \frac{12x(1-x)}{n(n+2)(n+3)}\left[\left(1 - \frac{10}{n+4}\right)x(1-x) + \frac{2}{n+4}\right] - \frac{x(1-x)}{n}\left[x^3(1-x)^n + (1-x)^3x^n\right],$$

from which inequality (10) follows. ∎

4. Proofs

Proof of Theorem 1. Let $x \in (0,1)$ be a fixed point. For the measure $\nu \in \mathbb{M}$ there exists a uniquely determined function F of bounded variations such that

$$\int_0^1 g\,dF = \int_0^1 g\,d\nu, \qquad F(x) = 0$$

for all $g \in L^1_\nu$.

Using Lemma 1 for this function F we get that

$$\begin{aligned}(D_n\nu)(x) &= (n+1)\left[F(1)x^n - F(0)(1-x)^n\right] \\ &\quad - n(n+1)\sum_{k=0}^{n-1}\frac{N_{k+1,n+1}(x)}{x(1-x)}\left(x - \frac{k+1}{n+1}\right) \\ &\quad \times\left(\int_0^1 N_{k,n-1}(t)F(t)\,dt\right) \\ &= A_n(x) - B_n(x). \end{aligned} \tag{16}$$

For the first term on the right-hand side of (16) we have

$$\begin{aligned}|A_n(x)| &= (n+1)\left|F(1)x^n - F(0)(1-x)^n\right| \\ &= (n+1)\left|(F(1) - F(x))x^n\right| - \left|(F(0) - F(x))(1-x)^n\right| \\ &\leqq \nu^*(x)(n+1)\left[x(1-x)^n + (1-x)x^n\right].\end{aligned}$$

From the well-known identity

$$\sum_{k=0}^{n+1}\binom{n+1}{k}x^k(1-x)^{n+1-k} = 1$$

we have

$$(n+1)\left[x(1-x)^n + (1-x)x^n\right] \leqq 1 \qquad (x \in [0,1];\, n \in \mathbb{N}),$$

thus we obtain that

$$|A_n(x)| \leqq \nu^*(x) \qquad (x \in [0,1];\, n \in \mathbb{N}). \tag{17}$$

Let us consider the second term on the right-hand side of (16). Since $F(x) = 0$ thus

$$\begin{aligned}|B_n(x)| &= n(n+1)\left|\sum_{k=0}^{n-1} \frac{N_{k+1,n+1}(x)}{x(1-x)}\left(x - \frac{k+1}{n+1}\right)\int_0^1 N_{k,n-1}(t)\right. \\ &\qquad\qquad \left.\times [F(t) - F(x)]\, dt\right| \\ &\leqq \nu^*(x)\frac{n(n+1)}{x(1-x)}\sum_{k=0}^{n-1} N_{k+1,n+1}(x)\left|x - \frac{k+1}{n+1}\right| \\ &\qquad \times \int_0^1 N_{k,n-1}(t)|t-x|\, dt.\end{aligned}$$

Applying the Cauchy inequality and the fundamental identity

$$\sum_{k=0}^{n+1} N_{k,n+1}(x)\left(x - \frac{k}{n+1}\right)^2 = \frac{x(1-x)}{n+1} \qquad (n \in \mathbb{N})$$

we conclude that

$$\begin{aligned}|B_n(x)| &\leqq \nu^*(x)\frac{n(n+1)}{x(1-x)} \\ &\qquad \times \left[\sum_{k=0}^{n-1} N_{k+1,n+1}(x)\left(x - \frac{k+1}{n+1}\right)^2\right]^{1/2} \\ &\qquad \times \left[\sum_{k=0}^{n-1} N_{k+1,n+1}(x)\left(\int_0^1 N_{k,n-1}(t)|t-x|\, dt\right)^2\right] \\ &\leqq \nu^*(x)\frac{n\sqrt{n+1}}{\sqrt{x(1-x)}} \\ &\qquad \times \left[\sum_{k=0}^{n-1} N_{k+1,n+1}(x)\left(\int_0^1 N_{k,n-1}(t)|t-x|\, dt\right)^2\right]^{1/2}.\end{aligned}$$

Using the Cauchy inequality with respect to the integrals and also (12) we have

$$|B_n(x)| \leqq \nu^*(x) \frac{\sqrt{n(n+1)}}{\sqrt{x(1-x)}} \times \left[\sum_{k=0}^{n-1} N_{k+1,n+1}(x) \int_0^1 N_{k,n-1}(t)(t-x)^2 \, dt \right]^{1/2}.$$

Finally, by (9) we can write

$$|B_n(x)| \leqq \sqrt{2}\,\nu^*(x) \qquad (x \in (0,1);\, n \in \mathbb{N}). \tag{18}$$

The statement of Theorem 1 immediately follows from (16)–(18). ∎

From Theorem 1 and (2) immediately follows

COROLLARY 1.

(i) For every measure $\nu \in \mathbb{M}$ we have

$$D^*\nu \in L^0.$$

(ii) The maximal operator $D^*\colon \mathbb{M} \to L^0$ is of weak type, i.e., the inequality

$$|\{x \in [0,1] \colon (D^*\nu)(x) > y\}| \leqq \frac{5(\sqrt{2}+1)}{y} \|\nu\|$$

holds for all $y > 0$ and all $\nu \in \mathbb{M}$.

Now we show that the result of M. M. Derrienic (see Theorem A) follows from Theorem 1, too.

COROLLARY 2. *Let $\nu \in \mathbb{M}$ be an absolutely continuous measure. Denote f as the Radon–Nikodym derivative of ν. Then*

$$\lim_{n\to\infty} D_n\nu = f \qquad a.e. \text{ on } [0,1]. \tag{19}$$

Proof. Since for the absolutely continuous measure ν we have $D_n\nu = D_n f$, it is enough to prove that

$$\lim_{n\to\infty} D_n f = f \qquad \text{a.e. on } [0,1] \tag{20}$$

for all $f \in L^1$.

Let m be a fixed natural number and consider the polynomial $f(x) = x^m$ ($x \in [0,1]$). M. M. Derrienic proved (see [5, Proposition I.2]) that for $m \leqq n$

$$D_n f(x) = \frac{(n+1)!}{(n+m+1)!} \sum_{r=0}^{m} \binom{m}{r} \frac{m!}{r!} \frac{n!}{(n-r)!} x^r. \tag{21}$$

It is easy to see that the main coefficient of (21) tends to 1 if $n \to \infty$ and the other coefficients of (21) tend to 0 if $n \to \infty$. This means that the limit relation (20) is satisfied for all polynomials.

Thus the statement follows from Corollary 1 by standard argument (see [16, p. 81]). ∎

Proof of Theorem 2. Let $\nu \in \mathbb{M}$ be a finite Borel measure on the interval [0, 1]. Consider the Lebesgue decomposition of ν,

$$\nu = \nu_f + \lambda,$$

where ν_f is an absolutely continuous measure and λ is a singular measure.

Since $D_n \nu = D_n \nu_f + D_n \lambda$ thus according to Corollary 2 for the proof of Theorem 2 it remains to establish that for every singular measure λ

$$\lim_{n \to \infty} D_n \lambda = 0 \qquad \text{a.e. on } [0,1]. \tag{22}$$

Let us consider a singular function $F\colon [0,1] \to \mathbb{R}$ with the property

$$\int_A dF = \int_A d\nu, \qquad A \subseteq [0,1].$$

From Lemma 1 we have

$$\begin{aligned} D_n(dF)(x) &= (n+1)\left[F(1)x^n - F(0)(1-x)^n\right] \\ &\quad - n(n+1) \sum_{k=0}^{n-1} \frac{N_{k+1,n+1}(x)}{x(1-x)} \\ &\quad \times \left(x - \frac{k+1}{n+1}\right) \int_0^1 N_{k,n-1}(t) F(t)\, dt. \end{aligned}$$

Using the identity

$$\sum_{k=0}^{n+1} N_{k,n+1}(x) \left(x - \frac{k}{n+1}\right) = 0 \qquad (x \in [0,1];\ n \in \mathbb{N})$$

we obtain that

$$D_n(dF)(x) = (n+1)[F(1) - F(x)]x^n$$
$$+(n+1)[F(x) - F(0)](1-x)^n$$
$$- n(n+1)\sum_{k=0}^{n-1} \frac{N_{k+1,n+1}(x)}{x(1-x)}\left(x - \frac{k+1}{n+1}\right)$$
$$\times \int_0^1 N_{k,n-1}(t)[F(t) - F(x)]\,dt. \tag{23}$$

It is obvious that for every $x \in (0,1)$ the first two terms of the right-hand side of (23) tend to 0 if $n \to \infty$.

Since F is a singular function, for almost every $x \in (0,1)$ we have

$$\lim_{t \to x} \frac{F(t) - F(x)}{t - x} = 0.$$

Fix a point x with the above property and $\varepsilon > 0$. Then there exists a number $\delta > 0$ such that

$$|F(t) - F(x)| < \varepsilon |t - x| \tag{24}$$

for $|t - x| < \delta$, $t \in (0,1)$.

The remainder of the right-hand side of (23) can be written in the form

$$-C_n(x) := n(n+1)\sum_{k=0}^{n-1} \frac{N_{k+1,n+1}(x)}{x(1-x)}\left(x - \frac{k+1}{n+1}\right)$$
$$\times \int_0^1 N_{k,n-1}(t)[F(t) - F(x)]\,dt$$
$$= n(n+1)\sum_{k=0}^{n-1} \frac{N_{k+1,n+1}(x)}{x(1-x)}\left(x - \frac{k+1}{n+1}\right)$$
$$\times \int_{|t-x|<\delta} N_{k,n-1}(t)[F(t) - F(x)]\,dt$$
$$+ n(n+1)\sum_{k=0}^{n-1} \frac{N_{k+1,n+1}(x)}{x(1-x)}\left(x - \frac{k+1}{n+1}\right)$$
$$\times \int_{|t-x|>\delta} N_{k,n-1}(t)[F(t) - F(x)]\,dt$$
$$=: I_1 + I_2. \tag{25}$$

Similarly to the proof of (18), we can conclude from (24) that

$$|I_1| \leqq \varepsilon \frac{n(n+1)}{x(1-x)} \sum_{k=0}^{n-1} N_{k+1,n+1}(x) \left| x - \frac{k+1}{n+1} \right| \int_0^1 N_{k,n-1}(t)|t-x|\,dt$$
$$\leqq \sqrt{2}\,\varepsilon. \tag{26}$$

For the term I_2 we obtain

$$|I_2| \leqq \frac{2M}{\delta^2} \frac{n(n+1)}{x(1-x)} \sum_{k=0}^{n-1} N_{k+1,n+1}(x) \left| x - \frac{k+1}{n+1} \right|$$
$$\times \int_0^1 N_{k,n-1}(t)|t-x|^2\,dt,$$

where $|F(t)| \leqq M$ $(t \in (0,1))$. Using the method of the proof of (18) we can conclude that

$$|I_2| \leqq \frac{2M}{\delta^2} \sqrt{\frac{n(n+1)}{x(1-x)}} \left(\sum_{k=0}^{n-1} N_{k+1,n+1}(x) \int_0^1 N_{k,n-1}(t)(t-x)^4\,dt \right)^{1/2}.$$

Thus from Lemma 2 we get

$$|I_2| \leqq \frac{18M}{\delta^2} \frac{\sqrt{n+1}}{n}. \tag{27}$$

From (25)–(27) it follows that for large enough $n \in \mathbb{N}$ we have

$$|C_n(x)| \leqq \varepsilon \qquad \text{a.e. } [0,1]$$

which proves (22) and this completes the proof of the theorem. ∎

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